Ruslan Olhovych | Test preparation materials | Mathematicats

Infinite nesting

A usual way to deal with infinitely nested expressions is to set a repeating part equal to some parameter, form an equation with respect to that parameter and solve it.

For example, to calculate $\sqrt{2\sqrt{2\sqrt{2\sqrt{\dots}}}}$ first we set $\sqrt{2\sqrt{2\sqrt{2\sqrt{\dots}}}}=t$. From that we get an equation $t=\sqrt{2t}$, whose solutions are 0 and 2. Zero is obviously not a reasonable root, so two must be the actual value of the considered nested root.

Exercises

№1. Calculate $\sqrt{3\sqrt{5\sqrt{3\sqrt{\dots}}}}$.

$\sqrt[3]{45}$

№2. Solve an equation $x^{x^{x^{\dots}}}=2$.

$\sqrt{2}$

№3. *(MIT Integration Bee, 2020)* Calculate $\int\sqrt{x\sqrt{x\sqrt{x\sqrt{\dots}}}}\,\mathrm{d}x$.

$\frac{x^2}{2}$

№4. *(MIT Integration Bee, 2014)* Calculate $\int\limits_0^2 \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{\dots}}}}\,\mathrm{d}x$.

$\boxed{\frac{19}{6}}$

№5. *(MIT Integration Bee, 2010)* Calculate $\int\limits_0^1 \sqrt{1+x\sqrt{1+x\sqrt{1+x\sqrt{\dots}}}}\,\mathrm{d}x$.

$\boxed{\frac{\varphi}{2}+\arctan\varphi}$

№6. *(MIT Integration Bee, 2026)* Calculate $\int\limits_0^1 \sqrt{x^2+x+\sqrt{x^2+x+\sqrt{\dots}}}\,\mathrm{d}x$.

$\boxed{\frac{3}{2}}$

№7. *(MIT Integration Bee, 2024)* Calculate $\int\limits_1^3 \frac{x+\frac{x+\dots}{1+\dots}}{1+\frac{x+\dots}{1+\dots}}\,\mathrm{d}x$.

$\boxed{2\sqrt{3}-\frac{2}{3}}$

№8. *(MIT Integration Bee, 2018)* Calculate $\int \sqrt{x\sqrt[3]{x\sqrt[4]{x\sqrt[5]{\dots}}}}\,\mathrm{d}x$.

$\boxed{\frac{x^{e-1}}{e-1}}$

№9. Prove that so-called *golden ratio* $\varphi = \dfrac{1+\sqrt{5}}{2}$ can be expanded as the following infinite continued fraction:

$$\varphi = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}$$

№10. *(MIT Integration Bee, 2017)* Calculate $\lim\limits_{n\to\infty} I_n$, where $I_1 = \int\limits_0^1 \dfrac{\mathrm{d}x}{1+\sqrt{x}}$, $I_2 = \int\limits_0^1 \dfrac{\mathrm{d}x}{1+\frac{1}{1+\sqrt{x}}}$, $I_3 = \int\limits_0^1 \dfrac{\mathrm{d}x}{1+\frac{1}{1+\frac{1}{1+\sqrt{x}}}}$, ...

$\boxed{\frac{\sqrt{5}-1}{2}}$

№11. *(MIT Integration Bee, 2025)* Calculate $\int \sqrt[3/1]{x\sqrt[4/2]{x\sqrt[5/3]{x\sqrt[6/4]{\ldots}}}}\,\mathrm{d}x$.

$\boxed{\frac{x^2}{2}}$