Ruslan Olhovych | Test preparation materials | Mathematicats

On One Classical Change of Variable

Often in textbooks you can encounter equations of the form

$$aR(x) + \frac{b}{R(x)} = c,$$

which are easily transformed to quadratic by change $R(x) = t$:

$$at + \frac{b}{t} = c \longrightarrow at^2 - ct + b = 0.$$

Let us take a look at an example from one famous Ukrainian textbook for high-school students:

$$\frac{x-3}{x^2+4x+9} + \frac{x^2+4x+9}{x-3} = -2. \tag{1}$$

Note that the second term on LHS is turned over first term, thus after a change

$$\frac{x-3}{x^2+4x+9} = t$$

equation (1) transforms into

$$t + \frac{1}{t} = -2.$$

Multiplying both sides by t get

$$t^2 + 1 = -2t,$$

which after completion the square yields $t = -1$. So

$$\frac{x-3}{x^2+4x+9} = -1,$$

which as well reduces to a quadratic equation with roots -3 and -2. Also do not forget to make sure that these roots do not turn neither $x - 3$ nor $x^2 + 4x + 9$ into zero!

Exercises

№1. $(x+2)^2 + \dfrac{24}{x^2+4x} = 18.$

$-6;\ -2 \pm \sqrt{6};\ 2$

№2. $\dfrac{x^2+x-5}{x} + \dfrac{3x}{x^2+x-5} + 4 = 0.$

$-5;\ -1 \pm \sqrt{6};\ 1$

№3. $\dfrac{x^2+1}{x} + \dfrac{x}{x^2+1} = 2.9.$

$0.5;\ 2$

№4. $x^4 - \dfrac{50}{2x^4-7} = 14.$

$\pm 2;\ \pm \sqrt[4]{1.5}$

№5. $\sqrt[7]{\dfrac{5-x}{x+3}} + \sqrt[7]{\dfrac{x+3}{5-x}} = 2.$

1

№6. $\sqrt[2025]{\dfrac{x+5}{8-x}} + \sqrt[2025]{\dfrac{8-x}{x+5}} = 2.$

1.5

№7. $\dfrac{21}{x^2-4x+10} - x^2 + 4x = 6.$

$1;\ 3$

№8. $\dfrac{x-a}{x-b} + \dfrac{x-b}{x-a} = 2.5.$

$2a-b;\ 2b-a$

№9. $x^3 - x^2 - \dfrac{8}{x^3 - x^2} = 2.$

$\boxed{-1; 2}$

№10. $3^x - 9 \cdot 3^{-x} - 8 = 0.$

$\boxed{2}$

№11. $6^{x+2} - 6^{-x} - 35 = 0.$

$\boxed{0}$

№12. $4^x + 0.25^x = 4.25.$

$\boxed{\pm 1}$

№13. $\log_5 x + \log_x 25 = \cot^2 \dfrac{25\pi}{6}.$

$\boxed{5; 25}$

№14. $|\tan x + \cot x| = 2.$

$\boxed{-\frac{\pi}{2} \pm \frac{\pi}{4} + \pi n,\ n \in \mathbb{Z}}$

№17. Prove that so-called *golden ratio* $\varphi = \dfrac{1+\sqrt{5}}{2}$ can be expanded as the following infinite continued fraction:

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$$

№15. *(TMUA, 2023)* The difference between the maximum and minimum values of the function $f(x) = a^{\cos x}$, where $a > 0$ and x is real, is 3. Find the sum of the possible values of a.

$\boxed{\sqrt{13}}$

№16. *(TMUA, 2022)* Given that

$$\left(a^3 + \frac{2}{b^3}\right)\left(\frac{2}{a^3} - b^3\right) = \sqrt{2}$$

where a and b are real numbers, what is the least value of ab?

$-\sqrt{2}$