TMUA 2023

Paper 1

№1. Given that

$$\int_0^1 (ax+b)\ \mathrm{d}x = 1$$

and

$$\int_0^1 x(ax+b)\ \mathrm{d}x = 1$$

find the value of $a+b$.

4

№2. The graphs of $y = x^2 + 5x + 6$ and $y = mx - 3$, where m is a constant, are plotted on the same set of axes. Given that the graphs do not meet, what is the complete range of possible values of m?

$(-1; 11)$

№3. For any integer $n \geq 0$

$$\int_n^{n+1} f(x)\ \mathrm{d}x = n+1.$$

Evaluate

$$\int_0^3 f(x)\ \mathrm{d}x + \int_1^3 f(x)\ \mathrm{d}x + \int_2^3 f(x)\ \mathrm{d}x + \int_4^3 f(x)\ \mathrm{d}x + \int_5^3 f(x)\ \mathrm{d}x.$$

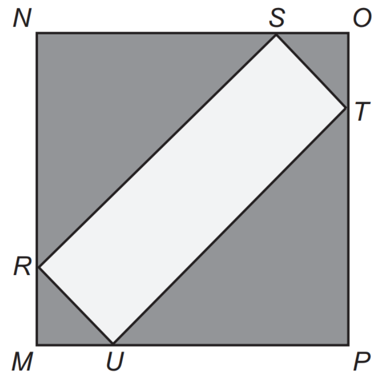
1

№4. Evaluate

$$\sum_{n=0}^{\infty} \frac{\sin\left(n\pi + \dfrac{\pi}{3}\right)}{2^n}.$$

$\frac{1}{\sqrt{3}}$

№5. The following shape has two lines of reflectional symmetry.



$MNOP$ is a square of perimeter 40 cm. The vertices of rectangle $RSTU$ lie on the edge of square $MNOP$. MR has length x cm. What is the largest possible value of x such that $RSTU$ has area 20 cm^2?

$5+\sqrt{15}$

№6. In the simplified expansion of $(2+3x)^{12}$, how many of the terms have a coefficient that is divisible by 12?

11

№7. $P(x)$ and $Q(x)$ are defined as follows:

$$P(x) = 2^x + 4,$$

$$Q(x) = 2^{2x-2} - 2^{x+2} + 16.$$

Find the largest value of x such that $P(x)$ and $Q(x)$ are in the ratio $4:1$, respectively.

$\log_2 12$

№8. A triangle XYZ is called *fun* if it has the following properties:

1. $\angle YXZ = 30°$;
2. $XY = \sqrt{3}a$;
3. $YZ = a$,

where a is a constant.

For a given value of a, there are two distinct fun triangles S and T, where the area of S is greater than the area of T.

Find the ratio

$$\text{area of } S : \text{area of } T.$$

$2:1$

№9. How many solutions are there to

$$(1 + 3\cos 3\theta)^2 = 4$$

in the interval $0° \leq \theta \leq 180°$?

5

№10. The trapezium rule with 4 strips is used to estimate the integral:

$$\int_{-2}^{2} \sqrt{4 - x^2}\, \mathrm{d}x.$$

What is the positive difference between the estimate and the exact value of the integral?

$2(\pi - 1 - \sqrt{3})$

№11. It is given that $f(x) = x^2 - 6x$. The curves $y = f(kx)$ and $y = f(x\breve{\ }c)$ have the same minimum point, where $k > 0$ and $c > 0$. Express k in terms of c.

$1 - \frac{c}{3}$

№12. How many solutions are there to the equation

$$\frac{2^{\tan^2 x}}{4^{\sin^2 x}} = 1$$

in the range $0 \leq x \leq 2$?

7

№13. Point P lies on the circle with equation $(x-2)^2 + (y-1)^2 = 16$. Point Q lies on the circle with equation $(x-4)^2 + (y+5)^2 = 16$. What is the maximum possible length of PQ?

$8 + 2\sqrt{2}$

№14. The function $f(x) = \frac{2}{3}x^3 + 2mx^2 + n$, $m > 0$ has three distinct real roots. What is the complete range of possible values of n, in terms of m?

$-\frac{8}{3}m^3 < n < 0$

№15. The difference between the maximum and minimum values of the function $f(x) = a^{\cos x}$, where $a > 0$ and x is real, is 3. Find the sum of the possible values of a.

$\sqrt{13}$

№16. A right-angled triangle has vertices at $(2, 3)$, $(9, ˘1)$ and $(5, k)$. Find the sum of all the possible values of k.

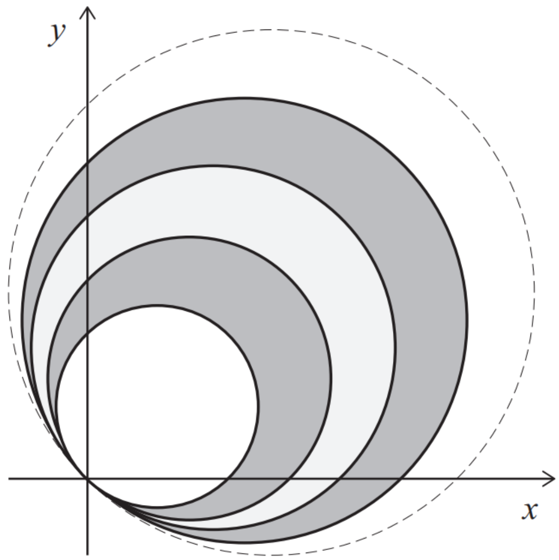
2.25

№17. A circle C_n is defined by

$$x^2 + y^2 = 2n(x + y)$$

where n is a positive integer.

C_1 and C_2 are drawn and the area between them is shaded. Next, C_3 and C_4 are drawn and the area between them is shaded. This is shown in the diagram.



This process continues until 100 circles have been drawn. What is the total shaded area?

10100π

№18. You are given that

$$S = 4 + \frac{8k}{49} + \frac{16k^2}{49} + \frac{32k^3}{343} + \ ... + 4 \cdot \left(\frac{2k}{7}\right)^n + \ ...$$

The value for k is chosen as an integer in the range $-5 \leq k \leq 5$. All possible values for k are equally likely to be chosen. What is the probability that the value of S is a finite number greater than 3?

$\frac{5}{11}$

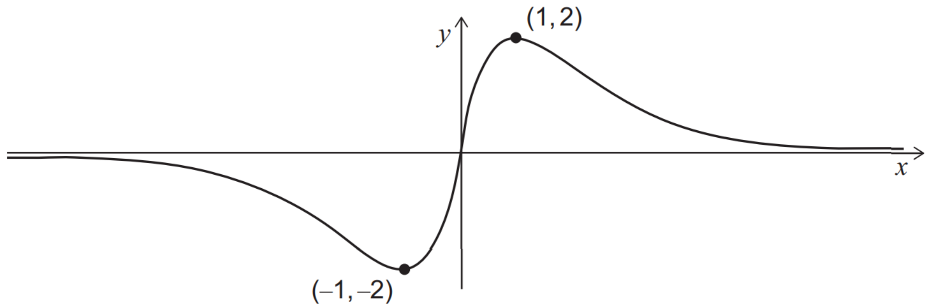
№19. The solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = |-6x| \quad \text{for all } x.$$

is $y = f(x) + c$, where c is a constant. Find $f(x)$.

$3x|x|$

№20. The diagram shows the graph of $y = f(x)$. The function f attains its maximum value of 2 at $x = 1$, and its minimum value of -2 at $x = -1$.



Find the difference between the maximum and minimum values of $(f(x))^2 - f(x)$.

6.25